WEST BENGAL STATE UNIVERSITY

B.Sc. Honours/Programme 3rd Semester Examination, 2022-23

MTMHGEC03T/MTMGCOR03T-MATHEMATICS (GE3/DSC3)

Time Allotted: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any *five* questions: $2\times5 = 10$

(a) State the Archimedean property of $\mathbb{R}$.

(b) Find the cluster points of the set

$$S=\{1,-1,1\tfrac{1}{2},-1\tfrac{1}{2},1\tfrac{1}{3},-1\tfrac{1}{3},\cdots\}.$$

(c) Find the greatest lower bound of the set $S=\left\{\frac{5}{n}:n\in\mathbb{N}\right\}$.

(d) Evaluate $\lim\limits_{n\to\infty}\left\{\frac{1^3}{n^4}+\frac{2^3}{n^4}+\frac{3^3}{n^4}+\cdots+\frac{n^3}{n^4}\right\}$.

(e) Test the convergence of the series $\sum\limits_{n=1}^{\infty}\frac{1}{\sqrt{n(n+1)}}$.

(f) Find the radius of convergence of

$$x+\frac{x^2}{2^2}+\frac{2!}{3^3}x^3+\frac{3!}{4^4}x^4+\cdots\cdots.$$

(g) Test the convergence of the series

$$\sum_{n=1}^{\infty}\sin\frac{1}{n}.$$

(h) Give an example of a Cauchy sequence with proper justification.

(i) Show that $\sum\limits_{n=1}^{\infty}\frac{\sin x}{n^2+n^4x^2}$ is uniformly convergent for all real x.

2. (a) If $x, y\in\mathbb{R}$ with $x>0$, $y>0$ then prove that there exists a natural number n such that $ny>x$. 4

(b) Let A be a non empty bounded above subset of $\mathbb{R}$. Let 4

$$B=\{-x:x\in A\}$$

Show that B is a non empty bounded below subset of $\mathbb{R}$ and

$$\inf B=-\sup A.$$

3. (a) Let A and B be subsets of $\mathbb{R}$ so that $A \subseteq B$. Let x be a cluster point of A. Show that x is a cluster point of B. 4

(b) Show that 1 and -1 are limit points of the set. 4

$$S = \left\{(-1)^m + \frac{1}{n} : m, n \in \mathbb{N}\right\}$$

4. (a) Justify that $\mathbb{Z}$ is a countable set. 4

(b) Show that the open interval (0, 1) is an uncountable set. 4

5. (a) Show that the sequence $\{x_n\}$ is monotone increasing, where 5

$$x_n = 1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \quad \text{for all } n \in \mathbb{N}$$

Hence show that the sequence $\{x_n\}$ is convergent.

(b) Apply Cauchy's criterion for convergence to show that the sequence $\left\{\frac{n}{n+1}\right\}$ is convergent. 3

6. (a) Show that the series 4

$$\sum_{n=1}^{\infty} \frac{3 \cdot 6 \cdot 9 \cdots (3n)}{7 \cdot 10 \cdot 13 \cdots (3n+4)} x^n$$

converges if $0 < x < 1$ and diverges if $x > 1$.

(b) Examine the convergence of the series 4

$$\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{2n+1}{n(n+1)}.$$

7. (a) Show that an absolutely convergent series is convergent. 4

(b) Give an example of a convergent series which is not absolutely convergent. 1

(c) Show that the sequence $\left\{\frac{n}{n+1}\right\}$ is a Cauchy sequence. 3

8. (a) Show that the sequence of functions $\{f_n\}$, where for all $n \in \mathbb{N}$, 4

$$f_n(x) = \frac{x^n}{1+x^n}, \quad x \geq 0$$

is pointwise convergent on $[0, \infty)$ but is not uniformly convergent on $[0, \infty)$.

(b) Examine uniform convergence of the sequence of functions $\{f_n\}$ on [0, 2], where for all $n \in \mathbb{N}$, 4

$$f_n(x) = \frac{x^n}{1+x^n}, \quad x \in [0, 2].$$

9. (a) Show that the series $\sum_{n=0}^{\infty}(1-x)x^n$ is not uniformly convergent on [0, 1]. 4

(b) With proper justification, show that $\lim_{x\to 0}\sum_{k=2}^{\infty}\frac{\cos kx}{k(k+1)}=\frac{1}{2}$. 4

10.(a) Find the radius of convergence of the power series 3

$$\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 5\cdot 8\cdots(3n-1)}x^n$$

(b) Use the fact that 5

$$\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n \qquad \forall\,|x|<1$$

to obtain the power series of $\frac{1}{(1+x)^3}$.

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